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Properties of perturbative solutions of unilateral matrix equations

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Abstract

A left-unilateral matrix equation is an algebraic equation of the form

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$$

where the coefficients a_r and the unknown x are square matrices of the same order and all coefficients are on the left (similarly for a right-unilateral equation). Recently certain perturbative solutions of unilateral equations and their properties have been discussed. We present a unified approach based on the generalized Bezout theorem for matrix polynomials. Two equations discussed in the literature, their perturbative solutions and the relation between them are described. More abstractly, the coefficients and the unknown can be taken as elements of an associative, but possibly noncommutative, algebra.

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1 Introduction

In a discussion of generalized Born-Infeld theories [1, 2] the construction of the Lagrangian was reduced to the solution of certain unilateral algebraic matrix equations and it was conjectured that the iterative solution of those equations is a sum of symmetric polynomials in the coefficients and of terms which are commutators. Equivalently, that the trace of the matrix solution is equal to the sum of traces of symmetric polynomials in the coefficients. The conjecture was later proven in [3] and by A. Schwarz in [4] using different methods and a slightly different form of the equation.

In the present note we combine Schwarz' s idea of expressing the trace of the solution as a contour integral in the complex plane of the trace of the resolvent of the corresponding matrix with the idea used in [3] of using the basic property of the trace of the logarithm of matrices.

The two approaches can be easily combined by using the generalized Bezout theorem for matrix polynomials. The qualitative algebraic fact that the trace of the solution is given by a sum of traces of symmetrized polynomials in the coefficients emerges almost without any computation. The coefficients in the expansion are not hard to compute and we give an explicit expression for them.¹ The method used here applies equally to the two different equations considered in [3] and [4]; in the last section we clarify the relation between the two equations and their solutions.

Before closing this section we recall the statement of the generalized Bezout theorem (see e.g. [5]). Let

$$P(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n, \quad \lambda \in \mathbb{C} \quad (1.1)$$

be a polynomial with square matrix coefficients a_r and x a square matrix of the same order. Define

$$P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \quad (1.2)$$

with the coefficients all on the left. It is easy to verify that

$$P(\lambda) - P(x) = Q(\lambda, x)(\lambda - x), \quad (1.3)$$

where

$$Q(\lambda, x) = \sum_{l=0}^{n-1} \lambda^l \left(\sum_{r=l+1}^n \epsilon a_r x^{r-l-1} \right). \quad (1.4)$$

¹Our contour integral formulas are very similar to Schwarz' s, but there seem to be some minor discrepancies between ours and his.

In other words $\lambda - x$ is a divisor of $P(\lambda) - P(x)$ on the right (if we had taken all the coefficients on the right, then $\lambda - x$ would have been a divisor on the left).

We shall study matrix equations of the type

$$P(x) = 0, \quad (1.5)$$

which we shall call left-unilateral matrix equations (meaning that all the coefficients are on the left). If x is a solution of (1.5), the characteristic polynomial $P(\lambda)$ of the equation can be factorized as

$$P(\lambda) = Q(\lambda, x)(\lambda - x). \quad (1.6)$$

It is clear that the Bezout theorem applies more abstractly if one considers a_r and x as elements of an associative, but possibly noncommutative, algebra. The same remark applies to the rest of this note, if one uses an appropriate algebraic definition of the trace as cyclic average (see [2]).

2 The Trace of the Solutions of Unilateral Matrix Equations

A. Schwarz [4] considers the unilateral matrix equation

$$x^n = 1 + \epsilon (a_0 + a_1 x + \dots + a_{n-1} x^{n-1}) . \quad (2.1)$$

where ϵ is a small parameter. For $\epsilon = 0$ the equation has n solutions in the complex plane, the n roots of 1. For $\epsilon \neq 0$ each of these solutions admits a perturbative expansion as a formal power series in ϵ . A. Schwarz proves that $\text{Tr } x^s$ can be expressed as a power series whose coefficients are symmetrized products of the a_i and gives an expression for the coefficients in terms of contour integrals in the complex plane.

We have found that a more explicit expression for $\text{Tr } x^s$ is provided by

$$\text{Tr } x^s = \text{Tr } 1 + s \sum_{k=1}^{\infty} \frac{\epsilon^k}{n^k} \sum_{n_0 + \dots + n_{n-1} = k} \frac{\text{Tr } \mathcal{S}(a_0^{n_0} \dots a_{n-1}^{n_{n-1}})}{n_0! \dots n_{n-1}!} \prod_{r=1}^{k-1} \left(s + \sum_{l=1}^{n-1} l n_l - r n \right). \quad (2.2)$$

This formula holds for positive as well as for negative values of s . Here the expansion is made around 1. A similar expansion could be derived for all the

n iterative solutions of the equation. The normalization of the symmetrized product $\mathcal{S}(a_0^{n_0} \dots a_n^{n_n})$ is chosen in such a way as to give the ordinary product if the factors commute [3].

If we introduce the notation

$$\binom{\alpha}{k} = \begin{cases} \prod_{r=1}^k \frac{\alpha - r + 1}{r} & \text{for } k = 1, 2, \dots \\ 1 & \text{for } k = 0, \end{cases} \quad (2.3)$$

which reduces to the usual definition of $\binom{\alpha}{k}$ for $\alpha \in \mathbb{N}$, then the result (2.2) can be rewritten

$$\begin{aligned} \text{Tr } x^s = & \quad (2.4) \\ \text{Tr } 1 + \frac{s}{n} \sum_{k=1}^{\infty} \epsilon^k (k-1)! \sum_{n_0 + \dots + n_{n-1} = k} \frac{\text{Tr } \mathcal{S}(a_0^{n_0} \dots a_{n-1}^{n_{n-1}})}{n_0! \dots n_{n-1}!} \binom{\frac{1}{n}(s - n + \sum_{l=1}^{n-1} l n_l)}{k-1}. \end{aligned}$$

As a first step to prove (2.2), we apply the generalized Bezout theorem. For (2.1) the characteristic polynomial is

$$P(\lambda) \equiv 1 - \lambda^n + \epsilon (a_0 + a_1 \lambda + \dots + a_{n-1} \lambda^{n-1}) \quad (2.5)$$

and, if $P(x) = 0$,

$$P(\lambda) = Q(\lambda, x)(\lambda - x) \quad (2.6)$$

with

$$Q(\lambda, x) = \sum_{l=0}^{n-1} \left(\sum_{r=l+1}^{n-1} \epsilon a_r x^{r-l-1} - x^{n-l-1} \right) \lambda^l. \quad (2.7)$$

Let x be one of the solutions of (2.1), e.g. the one which reduces to 1 for vanishing ϵ . (For the other solutions a similar procedure can be followed). We take the logarithm and then the trace of (2.6), and obtain

$$\text{Tr } \log P(\lambda) = \text{Tr } \log(Q(\lambda, x)) + \text{Tr } \log(\lambda - x). \quad (2.8)$$

Differentiation with respect to λ yields

$$\text{Tr } \frac{1}{P(\lambda)} P'(\lambda) = \text{Tr } \frac{1}{Q(\lambda, x)} Q'(\lambda, x) + \text{Tr } \frac{1}{\lambda - x}. \quad (2.9)$$

More generally, we can multiply the above equation by any function $f(\lambda)$ of λ , which is regular in a neighbourhood of 1, so as to get

$$\mathrm{Tr} \frac{1}{P(\lambda)} P'(\lambda) f(\lambda) = \mathrm{Tr} \frac{1}{Q(\lambda, x)} Q'(\lambda, x) f(\lambda) + \mathrm{Tr} \frac{1}{\lambda - x} f(\lambda). \quad (2.10)$$

As a next step, we follow Schwarz' s idea of using a contour integration to isolate the relevant part in the trace. We consider a small circle Γ around 1 in the complex plane, or more generally a small closed curve winding once around 1, and containing no other solution of (2.1) for $\epsilon = 0$. Then we compute the integral of equation (2.10) along it for small ϵ

$$\begin{aligned} (2\pi i)^{-1} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{1}{P(\lambda)} P'(\lambda) f(\lambda) = \\ (2\pi i)^{-1} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{1}{Q(\lambda, x)} Q'(\lambda, x) f(\lambda) + (2\pi i)^{-1} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{1}{\lambda - x} f(\lambda). \end{aligned} \quad (2.11)$$

The two integrals on the right hand-side can be evaluated through the Cauchy theorem in the following way. We are considering the case of small ϵ , and then Q is a polynomial in x and λ , which vanishes for λ near $e^{\frac{2\pi i k}{n}}$, $k = 1, \dots, n-1$, so that in this case its inverse Q^{-1} has no singularities near 1. Q' and f are regular functions near 1 and therefore the Cauchy theorem guarantees that the first term on the right hand-side vanishes:

$$(2\pi i)^{-1} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{1}{Q(\lambda, x)} Q'(\lambda, x) f(\lambda) = 0. \quad (2.12)$$

On the other hand $\mathrm{Tr} \frac{1}{\lambda - x} f(\lambda)$ has poles for λ near 1 with total residue $\mathrm{Tr} f(x)$. As x is close to 1 for small ϵ , the Cauchy theorem yields

$$(2\pi i)^{-1} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{1}{\lambda - x} f(\lambda) = \mathrm{Tr} f(x). \quad (2.13)$$

Finally, we obtain

$$\mathrm{Tr} f(x) = (2\pi i)^{-1} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{1}{P(\lambda)} P'(\lambda) f(\lambda). \quad (2.14)$$

Therefore, the problem of computing $\mathrm{Tr} f(x)$ amounts to evaluating the integral on the right hand-side of (2.14).

We can factorize

$$P(\lambda) = (1 - \lambda^n)T(\lambda) , \quad (2.15)$$

where

$$T(\lambda) = 1 - \epsilon(\lambda^n - 1)^{-1} \sum_{l=0}^{n-1} a_l \lambda^l . \quad (2.16)$$

Then

$$\frac{1}{P(\lambda)} P'(\lambda) = -\frac{n\lambda^{n-1}}{1 - \lambda^n} + \frac{1}{T(\lambda)} T'(\lambda) \quad (2.17)$$

and (2.14) becomes

$$\mathrm{Tr} f(x) = \frac{1}{2\pi i} \oint_{\Gamma} d\lambda \frac{n\lambda^{n-1}}{\lambda^n - 1} \mathrm{Tr} f(\lambda) + \frac{1}{2\pi i} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{1}{T(\lambda)} T'(\lambda) f(\lambda) . \quad (2.18)$$

The first integral on the right hand-side equals $\mathrm{Tr} f(1)$. To evaluate the second term we integrate by parts

$$\frac{1}{2\pi i} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{1}{T(\lambda)} T'(\lambda) f(\lambda) = -\frac{1}{2\pi i} \oint_{\Gamma} d\lambda \mathrm{Tr} \log(T(\lambda)) f'(\lambda) . \quad (2.19)$$

This is justified, because $\mathrm{Tr} \log T(\lambda)$ and $f(\lambda)$ are regular on Γ , so that

$$\frac{1}{2\pi i} \oint_{\Gamma} d\lambda \mathrm{Tr} \frac{d}{d\lambda} (\log(T(\lambda)) f(\lambda)) = 0. \quad (2.20)$$

In this way we obtain

$$\mathrm{Tr} f(x) = \mathrm{Tr} f(1) - \frac{1}{2\pi i} \oint_{\Gamma} d\lambda \mathrm{Tr} \log(T(\lambda)) f'(\lambda) . \quad (2.21)$$

We expand the logarithm

$$-\log(1 - \alpha) = \sum_{k=1}^{\infty} \frac{\alpha^k}{k} \quad \text{for } \alpha \text{ small} \quad (2.22)$$

and then make the following change of variable

$$y = \lambda^n . \quad (2.23)$$

This is possible, because if we start with a small closed curve Γ winding once around 1, then after the variable transformation (2.23) we still have a closed curve winding once around 1. We restrict ourselves to the case $f(\lambda) = \lambda^s$.

$$\begin{aligned}
& -\frac{s}{2\pi i} \oint_{\Gamma} d\lambda \operatorname{Tr} \log \left(1 - \epsilon \frac{\sum_{l=0}^{n-1} a_l \lambda^l}{\lambda^n - 1} \right) \lambda^{s-1} = \\
& s \sum_{k=1}^{\infty} \frac{\epsilon^k}{2\pi i k} \oint_{\Gamma} d\lambda \operatorname{Tr} \left(\frac{\sum_{l=0}^{n-1} a_l \lambda^l}{\lambda^n - 1} \right)^k \lambda^{s-1} = \\
& \frac{s}{n} \sum_{k=1}^{\infty} \frac{\epsilon^k}{2\pi i k} \oint_{\Gamma} dy \operatorname{Tr} \left(\frac{\sum_{l=0}^{n-1} a_l y^{\frac{l}{n}}}{y - 1} \right)^k y^{\frac{s}{n}-1}.
\end{aligned} \tag{2.24}$$

Notice that $y^{\frac{l}{n}}$ is regular in a neighbourhood of 1. In this form we can already see that the result is symmetrized in the coefficients $\{a_i\}$, because they enter through expressions of the type

$$\left(\sum_{l=0}^{n-1} a_l y^{\frac{l}{n}} \right)^k = \sum_{n_0 + \dots + n_{n-1} = k} \frac{k!}{n_0! \dots n_{n-1}!} \mathcal{S}(a_0^{n_0} \dots a_{n-1}^{n_{n-1}}) y^{\frac{1}{n} \sum_{l=1}^{n-1} l n_l}. \tag{2.25}$$

We obtain

$$\begin{aligned}
& \operatorname{Tr} x^s \\
& = \operatorname{Tr} 1 + \frac{s}{n} \sum_{k=1}^{\infty} \epsilon^k \sum_{n_0 + \dots + n_{n-1} = k} \frac{\operatorname{Tr} \mathcal{S}(a_0^{n_0} \dots a_{n-1}^{n_{n-1}})}{n_0! \dots n_{n-1}!} \frac{d^{k-1}}{dy^{k-1}} y^{\frac{1}{n} (\sum_{l=1}^{n-1} l n_l + s - n)} \Big|_{y=1} \\
& = \operatorname{Tr} 1 + s \sum_{k=1}^{\infty} \frac{\epsilon^k}{n^k} \sum_{n_0 + \dots + n_{n-1} = k} \frac{\operatorname{Tr} \mathcal{S}(a_0^{n_0} \dots a_{n-1}^{n_{n-1}})}{n_0! \dots n_{n-1}!} \prod_{r=1}^{k-1} \left(s + \sum_{l=1}^{n-1} l n_l - rn \right).
\end{aligned} \tag{2.26}$$

Here we have applied the Cauchy theorem in its more general form

$$(2\pi i)^{-1} \oint_C dy \frac{f(y)}{(y - y_0)^k} = \frac{1}{k-1!} \frac{d^{k-1}}{dy^{k-1}} f(y) \Big|_{y=y_0}, \tag{2.27}$$

where C a closed curve winding once around y_0 , and $f(y)$ is a function which is regular inside C .

In the simplest case $n = 2$ it is possible to solve the classical equation (i.e. the equation for commuting a_i) and this provides a closed expression for (2.2)

$$\mathrm{Tr} x = \mathrm{Tr} \left[\epsilon \frac{a_1}{2} + \mathcal{S} \sqrt{1 + \epsilon a_0 + (\epsilon \frac{a_1}{2})^2} \right], \quad (2.28)$$

where we again have chosen the solution which reduces to 1 for $\epsilon = 0$. In the appendix we show how expanding the square root in (2.28) near 1 we actually recover (2.2) in this case.

3 Relation between two unilateral matrix equations

The same procedure which we have used to prove (2.2), namely applying the generalized Bezout theorem and then the Cauchy theorem, can be equally well followed for the equation

$$\Phi = A_0 + A_1 \Phi + \dots A_n \Phi^n, \quad (3.1)$$

which is studied in [3]. If we define

$$A(\lambda) = A_0 + A_1 \lambda + \dots A_n \lambda^n \quad (3.2)$$

the characteristic polynomial of (3.1) is $\lambda - A(\lambda)$ and the result which corresponds to (2.21) is

$$\mathrm{Tr} f(\Phi) = -\frac{1}{2\pi i} \oint_C d\lambda \mathrm{Tr} \log(1 - \frac{A(\lambda)}{\lambda}) f'(\lambda). \quad (3.3)$$

Here C is a closed curve winding once around 0. We consider the particular case $f(\lambda) = \lambda^s$ and expand the logarithm

$$\mathrm{Tr} \Phi^s = s \sum_{k=1}^{\infty} \frac{(k-1)!}{2\pi i} \sum_{n_0+\dots+n_n=k} \frac{\mathrm{Tr} \mathcal{S}(A_0^{n_0} \dots A_n^{n_n})}{n_0! \dots n_n!} \oint_C d\lambda \lambda^{\sum_{l=0}^n (l-1)n_l + s-1}. \quad (3.4)$$

The Cauchy theorem has the effect of selecting the words of dimension $\sum_{l=0}^n (l-1)n_l = -s$ in the expansion of the logarithm, i.e.

$$\mathrm{Tr} \Phi^s = s \mathrm{Tr} \sum_{k=1}^{\infty} \frac{1}{k} (A_0 + \dots A_n)^k |_{\sum_{l=0}^n (l-1)n_l = -s} \quad (3.5)$$

and we recover the result already stated in [3], where the concept of dimension of a word was introduced. It should be remarked that the solution to (3.1) for vanishing coefficients is $\Phi = 0$. Consistent with that, as already noted in in [3], the result (3.5) only holds for positive integers s : it has no sense to invert the solution in this case, but see below.

To study more closely the relation between equation (2.1) and (3.1) we make the Ansatz

$$x = 1 + \alpha\Phi. \quad (3.6)$$

(Again, the same procedure could be followed for any of the roots of unity which are the solutions of (2.1) for $\epsilon = 0$.)

We choose the parameter α in such a way that

$$\alpha^{n-1} = -n. \quad (3.7)$$

There is no reason for α to be real, since most of the solutions of (2.1) are not real, even for real coefficients.

Then the relation between the n coefficients a_i and the $n + 1$ coefficients A_i is easily found to be

$$A_l = \begin{cases} -\alpha^{l-n}\epsilon \sum_{r=l}^{n-1} \binom{r}{l} a_r & \text{for } l = 0, 1 \\ \alpha^{l-n} \binom{n}{l} - \alpha^{l-n}\epsilon \sum_{r=l}^{n-1} \binom{r}{l} a_r & \text{for } 2 \leq l \leq n. \end{cases} \quad (3.8)$$

Some remarks can be made with respect to (3.8). First, observe that $A_n = 1$ is fixed, but this had to be expected, because the equation (2.1) has one coefficient less than the equation (3.1).

Moreover, (3.8) is a linear relation, and it is invertible, so that once the result that the trace of the powers of a solution depends only on the symmetrized products of the coefficients is proven for one of the two equations, it immediately follows also for the other. Negative powers of x can be expanded from (3.6) into a series of positive powers of Φ .

If ϵ is small, only the first two coefficients A_0 and A_1 are automatically small, the other coefficients satisfy

$$A_l \xrightarrow{\epsilon \rightarrow 0} \alpha^{l-n} \binom{n}{l} \quad \text{for } 2 \leq l \leq n. \quad (3.9)$$

Therefore, the two expansions (2.2) and (3.5) do not necessarily hold for the same range of the coefficients.

Appendix

In this appendix we check that (2.2) actually is the series expansion of (2.28) for $s = 1$, $n = 2$. We start by expanding the square root in (2.28) around 1

$$\begin{aligned} \text{Tr } x &= \text{Tr} \left[1 + \epsilon \frac{a_1}{2} + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{1}{2^r r!} (2r-3)!! \mathcal{S} \left(\epsilon a_0 + \left(\epsilon \frac{a_1}{2} \right)^2 \right)^r \right] \\ &= \text{Tr } 1 + \epsilon \frac{\text{Tr } a_1}{2} \\ &\quad + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{1}{2^r} (2r-3)!! \sum_{n_0=0}^r \frac{\epsilon^{2r-n_0}}{n_0! (r-n_0)!} \text{Tr } \mathcal{S} \left(a_0^{n_0} \left(\frac{a_1}{2} \right)^{2(r-n_0)} \right). \end{aligned} \quad (\text{A.1})$$

We introduce the new variables:

$$n_1 = 2(r - n_0), \quad k = n_0 + n_1 = 2r - n_0. \quad (\text{A.2})$$

Then we can rewrite

$$\begin{aligned} \text{Tr } x &= \text{Tr } 1 + \epsilon \frac{\text{Tr } a_1}{2} \\ &\quad + \sum_{k=1}^{\infty} \frac{\epsilon^k}{2^k} \sum_{\substack{n_0+n_1=k \\ n_1 \text{ even}}} (-1)^{\frac{n_1}{2}+n_0-1} \frac{(n_1-1)!!}{n_0! n_1!} (2n_0 + n_1 - 3)!! \text{Tr } \mathcal{S}(a_0^{n_0} a_1^{n_1}), \end{aligned} \quad (\text{A.3})$$

where we have used the relation

$$\frac{n_1!}{\frac{n_1}{2}! 2^{\frac{n_1}{2}}} = (n_1 - 1)!! \quad \text{for } n_1 \text{ even}. \quad (\text{A.4})$$

Due to the conditions $n_0 + n_1 = k$, n_1 even, the following relation holds

$$(-1)^{\frac{n_1}{2}+n_0-1} (n_1 - 1)!! (2n_0 + n_1 - 3)!! = \prod_{r=1}^{k-1} (1 + n_1 - 2r) \quad (\text{A.5})$$

and so formula (A.3) can be brought into the more compact form

$$\text{Tr } x = \text{Tr } 1 + \sum_{k=1}^{\infty} \frac{\epsilon^k}{2^k} \sum_{n_0+n_1=k} \frac{1}{n_0! n_1!} \text{Tr } \mathcal{S}(a_0^{n_0} a_1^{n_1}) \prod_{r=1}^{k-1} (1 + n_1 - 2r), \quad (\text{A.6})$$

which now coincides with (2.2) for $s = 1$, $n = 2$. It is no longer necessary to explicitly sum only over even values of n_1 , because

$$\prod_{r=1}^{k-1} (1 + n_1 - 2r) = 0 \quad \text{for } n_1 \text{ odd, } 1 \leq n_1 \leq k, \quad k > 1. \quad (\text{A.7})$$

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